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REMARKS TO THE PAPER: SWEEP ALGORITHM FOR SOLVING OPTIMAL CONTROL PROBLEM WITH MULTI-POINT BOUNDARY CONDITIONS

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Abstract. A new sweep algorithm for solving the discrete linear-quadratic problem (LQP) with three-point unseparated boundary conditions is proposed. This algorithm solves a more general problem than the algorithm proposed in [6]. We present an example which gives an accurate result and corrects technical inaccuracies missed in [3, 4].

Keywords: sweep algorithm, optimization, three-point boundary conditions, Euler-Lagrange equations.

AMS Subject Classification: 49J15, 49M25, 49N10.

1. Introduction

In [6], an algorithm was developed for solving LQP optimization with multipoint unseparated boundary conditions and in [3] a counterexample is given, in which shows that the results in [6] are non-optimal. However, scrupulous analysis showed that in [6] the case was considered where didn't take not all the nodal points (points in the boundary conditions) into account in the minimized quadratic functional (this is not mentioned in [6]). Note that, the statement of the problem is more general in [4], therefore, the example considered in [3, 4] cannot be solved directly using the sweep algorithm [6]. Therefore, in this paper, using the results of [4], a new sweep algorithm is given for solving the LQP optimization with three-point boundary conditions. Since the multipoint problem is more cumbersome and requires special investigation, we will be content with the LQP optimization of three-point problems. It is shown that, because of the technical error in [4], the final results of [3] are also not optimal, in spite of the fact that the minimum value of the functional is less than the one in [6]. Correcting these technical errors, we obtained the result for the example [3, 4], which is optimal.

2. Problem statement

In [6] the problem of LQP optimization with multipoint unseparated boundary conditions is considered. In this note, on the base of the algorithm [4], that increases the dimensionality of the original system, a new sweep method is proposed, which differs from the method in [6]. To develop this algorithm, we used the results of [4]. In order to avoid cumbersome calculations, we are satisfied with the three-point LQP optimization from [4, 6]

$$x\,(i+1) = \psi\,(i)\,x\,(i) + \Gamma\,(i)\,u\,(i)\,, \tag{1}$$

$$\Phi_1 x(0) + \Phi_2 x(s) + \Phi_3 x(l) = q, \tag{2}$$

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$$J = \sum_{i=0}^{l-1} \left(x'(i)\, R(i)\, x(i) + u'(i)\, C(i)\, u(i) \right) \to \min . \tag{3}$$

Here $x(i)$ is n-dimensional phase vector, $u(i)$ is m-dimensional vector of control influences, $\psi(i)$, $\Gamma(i)$ $(i = 0, 1, ..., l-1)$ are matricies of corresponding dimensions, being controllability pair [1, 2, 5], Φ_1, Φ_2, Φ_3 are constant matrices, such, that the system (2) satisfies the Kronekker-Kapelli condition [2, 4].

Using the corresponding Euler-Lagrange equations [1, 2, 4, 5] and the known formulas from [4, 6], the following algorithm can be proposed.

Step 1. The matrices $\psi(i)$, $\Gamma(i)$, Φ_1, Φ_2, $\Phi_3, q, R(i), C(i)$ are given and $M(i) = \Gamma(i)\, C^{-1} \Gamma'(i)$ are calculated.

Step 2. The iterations of matrices

$Q(i,j) = [E + (i.j)\, R(i+j)]^{-1}$,

$\psi(i,j) = \psi(i+j-1)\, Q(i,j-1)\, \psi(i, j-1)$, $\quad \psi(i,1) = \psi(i)$,

$M(i,j) = M(i+j-1) + \psi(i+j-1)\, Q(i,j-1)\, M(i,j-1)\, \psi'(i+j-1)$, $\quad M(i,1) = M(i)$,

$R(i,j) = R(i,j-1) + \psi'(i,j-1)\, R(i+j-1)\, Q(i,j-1)\, \psi(i,j-1)$, $\quad R(i,1) = R(i)$

are calculated.

Step 3. The matrices $H(s+1,\ l-s-1) = [E + M(s)\, R(s+1,\ l-s-1)]^{-1}$ are calculated.

Step 4. The following matrices

$$T_1(s) = R(s) + \psi'(s)\, R(s+1,\ l-s-1)\, H(s+1,\ l-s-1)\, \psi(s),$$
$$T_2(s) = \psi'(s)\, H'(s+1,\ l-s-1)\, \psi'(s+1,\ l-s-1)$$

are calculated .

Step 5. The following matrices

$$H_1(s) = \Phi_2 + \Phi_3 \psi(s+1, l-s-1)\, H(s+1, l-s-1)\, \psi(s),$$

$$H_2(s) = \Phi_3 \left[\psi(s+1, l-s-1)\, H(s+1, l-s-1)\, \mathrm{M}(s) + \psi'(s+1, l-s-1)\, \mathrm{M}(s+1, l-s-1) \right] \Phi_3$$

are calculated.

Step 6. The following matrices

$$K_1(s) = \Phi_1 + H_1(s)\, [E + M(0,s)\, T_1(s)]^{-1}\, \psi(0,s),$$

$$K_2(s) = -H_2(s) - H_1(s)\, [E + M(0,s)\, T_1(s)]^{-1}\, \mathrm{M}(0,s) \left(\Phi_2' + T_2(s)\, \Phi_3' \right),$$

$$K_3(s) = R(0,s) + \psi'(0,s)\, T_1(s)\, [E + M(0,s)\, T_1(s)]^{-1}\, \psi(0,s),$$

$$K_4(s) = \Phi_1 + \psi'(0,s) \left\{ E - T_1(s)\, [E + M(0,s)\, T_1(s)]^{-1}\, M(0,s) \right\} \left(\Phi_2' + T_2(s)\, \Phi_3' \right)$$

are calculated.

Step 7. The system of linear algebraic equations

$$\begin{bmatrix} K_1(s) & K_2(s) \\ K_3(s) & K_4(s) \end{bmatrix} \begin{bmatrix} x(0) \\ \nu \end{bmatrix} = \begin{bmatrix} q \\ 0 \end{bmatrix}$$

are solved and $x(0)$ and ν are found.

Step 8. Finally, using $x(0)$ and $\lambda(0) = \Phi_1' \nu$ and using the formulas

$$x(i+1) = \psi(i)\, x(i) + M(i) \psi'(i)^{-1} [-\lambda(i) + R(i) x(i)], \tag{4}$$

$$\lambda\,(i+1) = -\psi'\,(i)^{-1}\,[-\lambda(i) + R(i)x(i)], \tag{5}$$

first we find $x\,(1)$ and $\lambda\,(1)$, then under this scheme we alternately find the $x\,(i)$ and $\lambda\,(i)$ at $i = 2,\,...,s-1, s+1,\,...l-1$ and for $i = s$ the formulas

$$\lambda\,(s+1) = -\psi'\,(s)^{-1}\,[\Phi_2' v - \lambda(s) + R(s)x(s)], \tag{6}$$

$$x\,(s+1) = \psi\,(s)\,x(s) + M(s)\psi'\,(s)^{-1}\,[\Phi_2' v - \lambda(s) + R(s)x(s)] \tag{7}$$

are used.

Step 9. Then according to the formula

$$u\,(i) = -C^{-1}(i)\Gamma'\,(i)\,\lambda\,(i+1) \tag{8}$$

the control $u\,(i)$ is determined and, thus, the solution of the original problem (1) - (3) is found. From formulas (4) - (8) it can be seen that to find $x\,(i)$ and $u\,(i)$ we need to find $\lambda\,(i)$. However, the presentation of solutions $x\,(i)$ and $u\,(i)$ through $\lambda\,(i)$ is considered inexpedient and therefore we will try to exclude $\lambda\,(i)$ from expressions (4) and (8).

For this, using $\lambda\,(0) = \Phi_1'\nu$ and the formula (5), we found $\lambda\,(1)$, $\lambda\,(2)$ as follows:

$$\lambda\,(1) = -\psi'\,(0)^{-1}\,\Phi_1'\nu - \psi'\,(0)^{-1}\,R(0)x(0),$$

$$\lambda\,(2) = -\psi'\,(1)^{-1}\,\psi'\,(0)^{-1}\,\Phi_1'\nu - \psi'\,(1)^{-1}\,\psi'\,(0)^{-1}\,R(0)x(0) - \psi'\,(1)^{-1}\,R(1)x(1).$$

Next, using the method of induction , we obtain for $\lambda\,(i)$ the following expression:

$$\lambda\,(i) = \left[-\prod_{k=0}^{i-1}\psi'\,(i-1-k)^{-1}\right]\Phi_1'\nu - \sum_{k=0}^{i-1}\left[\prod_{j=k}^{i-1}\psi'\,(i-1-j-k)^{-1}\right]R(k)x(k). \tag{9}$$

Substituting expression (9) into (4) and (8), we obtain explicit expressions for $x\,(i)$ and $u\,(i)$, that do not depend on$\lambda\,(i)$:

$$\begin{aligned} x\,(i+1) = \psi\,(i)\,x\,(i) + \Gamma\,(i)\,\psi'\,(i)^{-1} \times \Bigg\{\left[-\prod_{k=0}^{i-1}\psi'\,(i-1-k)^{-1}\right]\Phi_1'\nu - \\ -\sum_{k=0}^{i-1}\left[\prod_{j=k}^{i-1}\psi'\,(i-1-j-k)^{-1}\right]R(k)x(k) + R(i)x(i)\Bigg\} \end{aligned} \tag{10}$$

at $i \neq s$ and $i = s+1$

$$\begin{aligned} x\,(s+1) = \psi\,(s)\,x\,(s) + \Gamma\,(s)\,\psi'\,(s)^{-1} \times \Bigg\{\Phi_2' v + \left[-\prod_{k=0}^{s-1}\psi'\,(s-1-k)^{-1}\right]\Phi_1'\nu - \\ -\sum_{k=0}^{s-1}\left[\prod_{j=k}^{s-1}\psi'\,(s-1-j-k)^{-1}\right]R(k)x(k) + R(s)x(s)\Bigg\}, \end{aligned} \tag{11}$$

and the control $u\,(i)$ is determined by the formula

$$u\,(i) = C^{-1}(i)\Gamma\,(i)\left\{\left[\prod_{k=0}^{i}\psi'\,(i-k)^{-1}\right]\Phi_1'\nu - \sum_{k=0}^{i}\left[\prod_{j=k}^{i}\psi'\,(i-j-k)^{-1}\right]R(k)x(k)\right\}. \tag{12}$$

To compare the methods proposed in this note and those in [4, 6], an example is considered

$n = 1,\ m = 1,\ l = 4,\ s = 2,\ \psi(0) = \psi(1) = 1,\ \psi(2) = \psi(3) = 2,$

where $\Gamma(0) = \Gamma(1) = \Gamma(2) = \Gamma(3) = 1, \quad \Phi_1 = \Phi_2 = \Phi_3 = 1, \quad q = 1,$

$R(0) = R(1) = R(2) = R(3) = 1, \quad C(0) = C(1) = C(2) = C(3) = 1.$

Based on the above algorithm, we obtained

$$x(0) = \frac{7}{58},\ x(1) = \frac{5}{58},\ x(2) = \frac{4}{29},\ x(3) = \frac{17}{58},\ x(4) = \frac{43}{58},$$
$$u(0) = -\frac{1}{29},\ u(1) = \frac{3}{58},\ u(2) = \frac{1}{58},\ u(3) = \frac{9}{58}.$$

Hence it is clear that the result ($J \approx 0.07$) obtained by the proposed method differs from the results of [3, 4]. As can be seen, the value of the functional (3) for this example is much less than the value of the functional for the solution in [3, 4] ($J \approx 0.5$) and in [6] ($J \approx 0.8$), which shows that the solutions of problem (1)-(3) obtained in [3, 4, 6] are not optimal. When analyzing the method described in [4], it turned out that because of a technical error, the resulting system of equations for finding the solution does not correspond to the Euler-Lagrange equations. Therefore, in spite of the fact that the solution obtained in [4] satisfies equation (1) and the boundary condition (2), it does not give the minimum value to the functional (3). Therefore the solution is not optimal.

It should be noted that sweep algorithm is proposed under an assumption of nonsingularity of $\psi(i)$. To remove this condition we need further investigation.

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